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BOUNDARY BEHAVIOR OF A NONPARAMETRIC SURFACE OF PRESCRIBED MEAN CURVATURE NEAR A REENTRANT CORNER

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ABSTRACT. Let Ω be an open set in $\mathbf{R}^2$ which is locally convex at each point of its boundary except one, say $(0,0)$. Under certain mild assumptions, the solution of a prescribed mean curvature equation on Ω behaves as follows: All radial limits of the solution from directions in Ω exist at $(0,0)$, these limits are not identical, and the limits from a certain half-space (H) are identical. In particular, the restriction of the solution to $\Omega \cap H$ is the solution of an appropriate Dirichlet problem.

0. Introduction. We consider here the behavior of a generalized solution of the equation for surfaces of prescribed mean curvature at an inner corner of the boundary where the solution is discontinuous. This work is a generalization of the previous work of the second author [**8**], which dealt with the minimal surface equation. It was shown there that all radial limits exist and that they are constant in directions coming from a half-space. Here we find that the same result holds for (nonparametric) surfaces of prescribed mean curvature.

1. Preliminaries. Let Ω be a bounded, open, connected, simply connected subset of $\mathbf{R}^2$ with $N = (0,0) \in \partial\Omega$ such that Ω is locally convex at each point of $\partial\Omega \setminus \{N\}$. Let $H(x,y,t)$ be a continuous function on $\Omega \times \mathbf{R}$ and $\phi \in C^0(\partial\Omega)$. We will make a number of assumptions which will hold throughout this work.

ASSUMPTIONS. (A) The equation

$$(p/W)_x + (q/W)_y = 2H(x,y,z(x,y)) \tag{1}$$

has a solution $z = f \in C^2(\Omega)$, where $p = z_x$, $q = z_y$, and $W^2 = 1 + p^2 + q^2$.

(B) $f \in C^0(\overline{\Omega} \setminus \{N\})$ and $f = \phi$ on $\partial\Omega \setminus \{N\}$.

(C) $F \notin C^0(\overline{\Omega})$.

(D) There exists $Q > 0$ such that $|H(x,y,f(x,y))| \leq Q$ for all $(x,y) \in \Omega$.

(E) The area of the graph of f over Ω is finite.

A function f satisfying the above conditions can be obtained as a generalized solution of the Dirichlet problem for (1) in Ω with data ϕ.

One method for obtaining such solutions is to minimize the functional

$$\iint_\Omega (1 + f_x + f_y)^{1/2}\,dx\,dy + \iint_\Omega \int_0^f 2H(x,y,t)\,dt\,dx\,dy + \int_{\partial\Omega} |f - \phi|$$

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in BV(Ω). Suppose that Ω has a locally Lipschitz boundary, H is Lipschitz and nondecreasing in t, and $H_0(x,y) = 2H(x,y,0)$ satisfies

$$\left|\iint_A H_0(x,y)\,dx\,dy\right| \leq (1-\varepsilon)\iint_A |D\chi_A| \tag{2}$$

for all Caccioppoli sets $A \subset \Omega$, where $0 < \varepsilon < 1$ is fixed and χ_A is the characteristic function of A. Then it is shown in [**4**] that there is a solution of (1) in Ω satisfying (E) (see also [**5**, Theorem 1.1]). Further, if $\partial\Omega \setminus \{N\}$ is smooth and its curvature $\kappa(x,y)$ satisfies

$$|2H(x,y,f(x,y))| \leq \kappa(x,y)$$

at each point, then (B) also holds.

On the other hand, if $H = H(x,y)$ and H has a bounded gradient, then the Perron method can be used as was done in [**9**] for H constant. (The essential ingredient necessary for the extension of Serrin's work to variable H is an interior gradient estimate and this is available in [**7** and **10**].) In order to use Perron's method, we need a bounded supersolution so that we know that the upper Perron class is nonempty. (A bounded subsolution would do just as well.) The results in [**5**] enable us to give sharp conditions on when such a supersolution can be found and also show that the two methods are closely related. If $H = H(x,y)$ is Lipschitz and Ω has a locally Lipschitz boundary, then Giusti shows that a necessary and sufficient condition for the existence of a solution of (1) in Ω is

$$\left|2\iint_A H(x,y)\,dx\,dy\right| < \iint |D\chi_A|$$

for all Caccioppoli sets A strictly contained in Ω with positive measure. If the strict inequality holds when $A = \Omega$, then it is shown that (2) holds. (In the boundary case $|2\int\int_\Omega H| = \int\int_\Omega |D\chi_\Omega| = H_1(\partial\Omega)$, the unique (up to a constant) solution of (1) is bounded iff $\kappa(x,y) < 2H(x,y)$.) We see then that in many cases the Perron method can be applied if and only if the variational method can.

Finally, if $H = H(x,y)$, using results from [**3**, §15], we can give domains Ω and boundary data ϕ such that (C) must hold. In particular, if N is an inner boundary point of Ω [**3**], then a bound for the value of $z(N)$, $z \in C^0(\overline{\Omega})$, satisfying (1) can be given which depends only on values of z at points of $\partial\Omega$ bounded away from N.

2. A parametric representation of $z = f(x,y)$. Define

$$S = \{(x,y,f(x,y)) | (x,y) \in \overline{\Omega} \setminus \{N\}\},$$
$$\Gamma = \{(x,y,\phi(x,y)) | (x,y) \in \partial\Omega\},$$

and $S_0 = S \setminus \Gamma$. Let $T = \{(0,0,z) | z \text{ is real}\}$ be the z-axis. We need some parameter domains, so set $E = \{(u,v) | u^2+v^2 < 1\}$, $B = \{(u,v) \in E | v > 0\}$, $\partial' B = \{(u,v) | u^2+v^2 = 1,\ v > 0\}$, $\partial'' B = \{(u,0) | -1 < u < 1\}$, and $B' = B \cup \partial' B$. Let $P = (0,0,\phi(0,0))$ and $\Gamma_0 = \Gamma \setminus \{P\}$.

LEMMA 2.1. *There is a vector $X \in C^0(B'\colon R^3) \cap C^2(B\colon R^3)$ with the following properties*:

(i) *X is a homeomorphism of B onto S_0.*
(ii) *X maps $\partial' B$ strictly monotonically onto Γ_0.*
(iii) *X is conformal on B, i.e. $X_u \circ X_v = 0$, $X_u^2 = X_v^2$ on B.*

(iv) $X_{uu} + X_{vv} = 2H(X)X_u \times X_v$ *on* B.

(v) *If we write* $X(u,v) = (x(u,v), y(u,v), z(u,v))$, *then* $x, y \in C^0(\overline{B})$ *and* $x(u,0) = y(u,0) = 0$ *for* $-1 \leq u \leq 1$.

(vi) *As* $0 \leq \theta \leq \pi$ *increases,* $K(\cos(\theta), \sin(\theta))$ *moves in a clockwise direction about* $\partial\Omega$. *Here* $K(u,v) = (x(u,v), y(u,v))$.

PROOF. At each point Q of S_0, there is a neighborhood U of Q in S_0 and a vector $Y \in C^2(E : \mathbf{R}^3)$ such that Y is a homeomorphism from E onto U, Y is conformal on E, and $\Delta Y = 2H(Y)Y_u \times Y_v$. Let us pick the vectors $\{Y\}$ so $\{Y_u \times Y_v\}$ gives a consistent orientation to S_0. We may now regard S_0 as a Riemann surface with local uniformizing parameters $\{Y^{-1}\}$. By the uniformization theorem for simply connected Riemann surfaces, there is a global uniformizing parameter Φ mapping E onto S_0. This means that Φ is a homeomorphism of E onto S_0 and that $Y^{-1} \circ \Phi$ is analytic for each local uniformizing parameter Y^{-1} when we regard the domains of Φ and Y as being subsets of the complex plane. Since $\Phi = Y \circ (Y^{-1} \circ \Phi)$ on an open subset of E, we see that Φ is conformal on E and that $\Delta\Phi = 2H(\Phi)\Phi_u \times \Phi_v$.

Set $\Phi(u,v) = (a(u,v), b(u,v), c(u,v))$ and consider the map $G(u,v) = (a(u,v), b(u,v))$. Notice that $\iint_E G_u^2 + G_v^2 \, du \, dv \leq 2 \iint_E |\Phi_u|^2 \, du \, dv = 2A(S) < \infty$, where $A(S)$ is the area of S. If we apply the proof of Theorem 2.4 of [**1**] to G, we see that $G \in C^0(\overline{E} : \mathbf{R}^2)$. If we define σ as the subset of ∂E which G maps onto $\partial\Omega \setminus \{N\}$, then $\Phi \in C^0(E \cup \sigma)$, since $f \in C^0(\overline{\Omega} \setminus \{N\})$.

Let $w_0 \in \partial\Omega$ and set $\Omega(\varepsilon) = \{w \in \Omega \mid |w - w_0| < \varepsilon\}$ and $E(\varepsilon) = G^{-1}(\Omega(\varepsilon))$ for all $\varepsilon > 0$. Since $G^{-1}(x,y) = \Phi^{-1}(x, y, f(x,y))$, G is a homeomorphism of E and Ω. For $\varepsilon > 0$ small enough, the open sets $\Omega(\varepsilon)$ and $\Omega \setminus \overline{\Omega(\varepsilon)}$ are connected and simply connected and so $E(\varepsilon)$ and $E \setminus \overline{E(\varepsilon)}$ are connected and simply connected. Thus $\phi \neq \partial E \cap \overline{E(\varepsilon)}$ is a connected arc of ∂E. Since $\{E(\varepsilon)\}$ is a nested collection, $\overline{E(\varepsilon)}$ converges to a closed, connected arc $\tau(w_0) \subset \partial E$ as $\varepsilon \to 0$. Notice that $\tau(w_0) = G^{-1}(w_0)$ by construction and so G is weakly monotonic on ∂E. In particular σ is connected and Φ maps σ weakly monotonically on Γ_0. Now if we use [**6**], we see that Φ maps σ strictly monotonically onto Γ_0.

If $\partial E \setminus \sigma$ were a single point, we could use the proof of Lemma 2.2 to show that $\Phi \in C^0(\overline{E})$ and so $f \in C^0(\overline{\Omega})$, in contradiction to (C). To finish, we need only compose Φ with a suitable conformal map of B onto E and obtain X. Q.E.D.

The proof of the next lemma is a (minor) modification of the proof of Lemma 2.2 of [**8**]. Our proof is self-contained for the sake of clarity.

LEMMA 2.2. $X \in C^0(\overline{B})$.

PROOF. We need prove only that $z \in C^0(\overline{B})$. We will prove that z is uniformly continuous on B and so extends to a function in $C^0(\overline{B})$.

Let $\varepsilon > 0$. Define $g(x,y)$ as the function whose graph is the upper half-sphere of radius $1/2Q$ centered at $(0,0,0)$. Pick $d > 0$ so that $g(0,0) - g(2d, 0) < \varepsilon/4$. For some $0 < \tau < \min(d, \varepsilon/4)$, the diameter of the shortest arc on $\Gamma \cup T$ joining two points on $\Gamma \cup T$ is less than $\min(d, \varepsilon/4)$ whenever the distance between the points is less than τ [**1**, p. 103]. Define $e(\delta) = 4A(S)/\ln(1/\delta)$ and pick $\delta > 0$ so that $2\pi e(\delta) < \tau^2$.

Let $w_0 = (u_0, v_0) \in B$. For $\delta \leq r \leq \sqrt{\delta}$, set $C_r = \{w \in \overline{B} \mid |w - w_0| = r\}$. Let (r, θ) be polar coordinates at w_0 and let $\xi(r,\theta) = X(w_0 + r(\cos(\theta), \sin(\theta)))$. If

$C_r \cap \partial B \neq \emptyset$, define $\alpha(r)$ and $\beta(r)$ by $0 \leq \alpha(r) < \beta(r) \leq 2\pi$ and $\{(r, \alpha(r)), (r, \beta(r))\} = C_r \cap \partial B$; otherwise $\alpha(r) = 0$ and $\beta(r) = 2\pi$. Set

$$p(r) = \int_{\alpha(r)}^{\beta(r)} |\xi_\theta(r,\theta)|^2 \, d\theta;$$

then

$$\int_\delta^{\sqrt{\delta}} p(r)/r \, dr \leq 2A(S).$$

For some $\rho \in [\delta, \sqrt{\delta}]$, $\ln(1/\delta)p(\rho)/2 \leq 2A(S)$ and so

$$p(\rho) \leq 4A(S)/\ln(1/\delta) = e(\delta).$$

If $C_\rho^* = X(C_\rho)$ and $L_\rho = \int_{\alpha(\rho)}^{\beta(\rho)} |\xi_\theta(\rho,\theta)| \, d\theta$ is the arclength of C'_ρ, then $L_\rho^2 \leq (\beta(\rho) - \alpha(\rho))p(\rho) \leq 2\pi e(\delta) < \tau^2$ and so $L_\rho < \tau < \min(d, \varepsilon/4)$. If $C_\rho \cap \partial B \neq \emptyset$, then the diameter of the shortest arc on $\Gamma \cup T$ joining the ends of C'_ρ is at most $\min(d, \varepsilon/4)$. For any $W \subset \overline{B}$, let $W^* = K(W)$.

Define J_ρ as the component of $B \setminus C_\rho$ which contains w_0. Set

$$m = \inf\{z(u,v) | (u,v) \in C_\rho \text{ or } (u,v) \in \partial' B \cap \overline{J}_\rho\}$$

and

$$M = \sup\{z(u,v) | (u,v) \in C_\rho \text{ or } (u,v) \in \partial' B \cap \overline{J}_\rho\}.$$

Let $U = K(J_\rho)$ and $D_\rho = C_\rho \cup (\overline{J}_\rho \cap \partial' B)$. Notice that $X(J_\rho)$ is the graph of f over U and that $\partial U \setminus D_\rho^*$ is either empty or contains the single point $(0,0)$. Now let us define $H'(x,y) = H(x,y,f(x,y))$ for $(x,y) \in \Omega$ and notice that $|H'(x,y)| \leq Q$ for all $(x,y) \in \Omega$. Notice that the diameter of U is less than $L_\rho + \operatorname{diam}(D_\rho^* \setminus C_\rho^*)$, so less than $2d$. Now we apply Lemma 2.2 of [**5**] (with "$\Omega = U$", "$u = M + g - g(2d,0)$", "$v = f$", and "$\Gamma_1 = \partial U \setminus \{N\}$") and see that

$$f(x,y) \leq M + g(x,y) - g(2d,0) \leq M + \varepsilon/4 \quad \text{for } (x,y) \in U.$$

Since $z(u,v) = f(x(u,v), y(u,v))$, we get $z(u,v) < M + \varepsilon/4$ for $(u,v) \in J_\rho$. Similarly, $-z(u,v) \leq -m + \varepsilon/4$ for $(u,v) \in J_\rho$. Thus $m - \varepsilon/4 < z(u,v) < M + \varepsilon/4$ for $(u,v) \in J_\rho$.

Now the diameter of $X(\partial' B \cap \overline{J}_\rho)$ is less than $\varepsilon/4$ and $L_\rho < \varepsilon/4$, so $M - m < \varepsilon/2$. If $(u,v) \in B$ and $|(u,v) - (u_0,v_0)| < \delta$, then $(u,v) \in J_\rho$ and so $|z(u,v) - z(u_0,v_0)| < M - m + \varepsilon/2 < \varepsilon$. Since δ is independent of w_0, z is uniformly continuous on B and so can be extended to $\overline{B}$ as a continuous function. Q.E.D.

3. Boundary behavior. By [**6**], $X \in C^1(B \cup \partial'' B)$ and the branch points of X on $\partial'' B$ are isolated. We see that

$$X_u(u,0) = (0,0,z_u(u,0)) \quad \text{and} \quad X_v(u,0) = (x_v(u,0), y_v(u,0), 0).$$

Let (r,θ) be polar coordinates centered at $(0,0)$. Then $\Omega = \{(r,\theta) | \alpha < \theta < \beta, 0 < r < r(\theta)\}$ for some $-\pi \leq \alpha < 0 < \beta \leq \pi$ (where we may need to rotate Ω about $(0,0)$). For each $\alpha < \theta < \beta$, set $x(t) = x(t,\theta) = t \cdot \cos(\theta)$ and $y(t) = y(t,\theta) = t \cdot \sin(\theta)$. We denote the radial limit of f at $(0,0)$ from the direction θ (if it exists) by

$$Rf(\theta) = \lim_{t \to 0+} f(x(t,\theta), y(t,\theta)).$$

Set $\lambda(t) = (x(t), y(t), f(x(t)), y(t))$ and $\omega(t) = X^{-1}(\lambda(t))$. Notice that $\lambda(t) \to (0, 0, Rf(\theta))$ as $t \to 0+$ if $\lambda(t)$ converges to a point as $t \to 0+$. Finally define $Rf(\alpha) = Rf(\beta) = \phi(0,0)$ and $u(\alpha) = -1$, $u(\beta) = 1$.

THEOREM 3.1. *For all $\alpha < \theta < \beta$, there is a unique $u(\theta) \in [-1,1]$ such that $\omega(t) \to (u(\theta), 0)$ as $t \to 0+$. Also, $u \in C^0([\alpha, \beta])$. Thus $Rf(\theta) = z(u(\theta), 0)$ exists for all $\alpha \le \theta \le \beta$ and $Rf \in C^0([\alpha, \beta])$.*

The proof is the same as the proof of Lemma 3.1 of [**8**]. The only facts we use in this proof are that $X \in C^1(B \cup \partial'' B)$, K is a homeomorphism of B and Ω, and the regular points of X on $\partial'' B$ are dense in $\partial'' B$. The fact that X maps $\partial' B$ (weakly or strongly) monotonically onto Γ_0 is not important and can be replaced by the following fact:

$$(x_v(u(\theta), 0), y_v(u(\theta), 0)) = |z_u(u(\theta), 0)|(\cos(\theta), \sin(\theta))$$

for $\alpha < \theta < \beta$ and so $u(\theta)$ is weakly monotonic on (α, β).

THEOREM 3.2. *There exist $\lambda \in [\alpha, \beta - \pi]$ and $u_0 \in [-1, 1]$ such that $u(\theta) = u_0$ for all $\theta \in [\lambda, \lambda + \pi]$; also X is strictly monotonic on $[-1, u_0]$ and on $[u_0, 1]$.*

PROOF. Suppose $\theta_0 \in [\beta - \pi, \alpha + \pi]$ and $X(u, 0)$ is not weakly monotonic on $-1 \le u \le u(\theta_0)$. Then X has a branch point at $(b, 0)$ for some $-1 < b < u(\theta_0)$ and $z_u(u, 0)$ changes sign at $u = b$, say $z_u(u, 0) > 0$ on $(b - \varepsilon, b)$ and $z_u(u, 0) < 0$ on $(b, b + \varepsilon)$ for some $\varepsilon > 0$. Pick a and c so that $b - \varepsilon < a < b < c < b + \varepsilon$ and $z(a, 0) = z(c, 0)$. Then $a = u(\theta(a))$, $c = u(\theta(c))$, and $\alpha < \theta(a) < \theta(c) < \alpha + \pi$. Let ω be a smooth Jordan arc in B from $(a, 0)$ to $(c, 0)$ such that $\sigma = X(\omega)$ is a simple, closed Jordan curve, $\tau = K(\omega)$ is a convex Jordan curve which is smooth except at $(0, 0)$, and the curvature $\kappa(x, y)$ satisfies $\kappa(x, y) \ge 2Q$ for all $(0, 0) \ne (x, y) \in \tau$. Let U be the open region bounded by τ. We will use the function $H'(x, y) = H(x, y, f(x, y))$ mentioned earlier.

First, let $h \in C^0(\overline{U} \setminus \{N\})$ be the unique variational solution of $(p/W)_x + (q/W)_y = H'(x, y)$ in U with $h = f$ on $\tau \setminus \{N\}$ [**4**]. (We know that (2) is satisfied because of [**5**].) If we use a barrier argument of Serrin [**9**, pp. 375–376] together with Lemma 2.2 of [**5**], we see that $h \in C^0(\overline{U})$.

Second, by Lemma 2.2 of [**5**], we see that $f = h$ on U. This implies that $Rf(\theta) = h(0, 0)$ for all $\theta \in (\theta(a), \theta(c))$, a contradiction. Thus $X(u, 0)$ is weakly monotonic on $-1 \le u \le u(\theta_0)$. Using [**6**], we see that $X(u, 0)$ is strictly monotonic on $-1 \le u \le u(\theta_0)$. Similarly, $X(u, 0)$ is strictly monotonic on $u(\theta_0) \le u \le 1$. If θ_1 is another element of $[\beta - \pi, \alpha + \pi]$, the same argument proves that $X(u, 0)$ is strictly monotonic on $-1 \le u \le u(\theta_1)$ and on $u(\theta_1) \le u \le 1$. Since $X(-1, 0) = X(1, 0) = P$, we see that $u(\theta_0) = u(\theta_1)$. Thus $u(\theta)$ is constant on $[\beta - \pi, \alpha + \pi]$.

Suppose now that $\theta_2, \theta_3 \in [\alpha, \beta]$ with $0 < \theta_3 - \theta_2 < \pi$. The argument above shows that X is strictly monotonic on $[u(\theta_2), u(\theta_3)]$. Q.E.D.

Let $\theta_L = \inf\{\theta \in [\alpha, \beta] | u(\theta) = u_0\}$ and $\theta_R = \sup\{\theta \in [\alpha, \beta] | u(\theta) = u_0\}$. We know that $\theta_R - \theta_L \ge \pi$. Also $u(\cdot)$ is a homeomorphism of $[\alpha', \theta_L]$ onto $[-1, u_0]$ and of $[\theta_R, \beta']$ onto $[u_0, 1]$, for some $\alpha \le \alpha' \le \theta_L$ and $\theta_R < \beta' \le \beta$. Thus

THEOREM 3.3. *The radial limits $Rf \in C^0([\alpha, \beta])$ behave as follows:*

(i) *The extreme values of $Rf(\theta)$ are $Rf(\alpha) = Rf(\beta) = \phi(0,0)$ and $Rf(0) = z(u_0, 0)$.*

(ii) $Rf(\theta)$ *is monotonic on* $[\alpha, \theta_L]$.
(iii) $Rf(\theta) = z(u_0, 0)$ *for all* $\theta \in [\theta_L, \theta_R]$.
(iv) $Rf(\theta)$ *is monotonic on* $[\theta_R, \beta]$.

Let U be open in $\mathbf{R}^2$ with $(0,0) \in \partial U$, $H(x,y,t)$ be continuous on $U \times \mathbf{R}$, $\phi \in C^0(\partial U)$, and f be a C^2 solution of (1) in U. Suppose that for some $\varepsilon > 0$, the set $\Omega = \{(x,y) \in U | x^2 + y^2 < \varepsilon^2\}$ is locally convex at every point of $\partial\Omega$ except $(0,0)$ and f satisfies assumptions (B), (D), (E) in Ω. Then f behaves at $(0,0)$ as indicated in Theorem 3.3. As in [**8**], we make the following

CONJECTURE. If $\theta_R - \theta_L > \pi$, then $f \in C^0(\overline{\Omega})$.

ADDED IN PROOF. The conjecture mentioned above has been proven in *Non-parametric minimal surfaces in* $\mathbf{R}^3$ *whose boundaries have a jump discontinuity* (preprint) by the second author.

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